A property of integer-valued Lyapunov function for a class of totally nonnegative matrices

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Abstract

In this paper, an integer-valued Lyapunov function $\sigma : \Lambda \to \{0, 1, 2, \cdots, n-1\}$ is first defined on a dense open subset Λ of $\mathbb{R}^n$. Subsequently, the integer-valued Lyapunov function is extended to $\mathbb{R}^n$ by defining $\sigma_M(x_0) = \max_{x \in U(x_0,\delta)\cap\Lambda} \sigma(x)$ for all $x_0 \in \mathbb{R}^n$. Then the property of the integer-valued Lyapunov function for a class of totally nonnegative matrices is investigated. And by using the method of classification, it is shown that $\sigma_M(Ax) \le \sigma(x)$ for all $x \in \Lambda$.
Key Words: Totally nonnegative matrix; Integer-valued Lyapunov function; Oscillatory matrix; Classification method

1 Introduction

In 1984, John Smillie[1] considered the following autonomous tridiagonal cooperative or competitive system:

$$\begin{cases} x_1' = f_1(x_1, x_2), \\ x_j' = f_j(x_{j-1}, x_j, x_{j+1}), 2 \le j \le n-1, \\ x_n' = f_n(x_{n-1}, x_n), \end{cases} \tag{1}$$

where $f = (f_1, f_2, \cdots, f_n)$ is defined on Ω, a nonempty open subset of $\mathbb{R}^n$. The f_i and their partial derivatives with respect to the x_j are assumed to be continuous, and further assume that there exist $\delta_i \in \{-1, 1\}, 1 \le i \le n-1$, such that $\delta_i \frac{\partial f_i}{\partial x_{i+1}} > 0, \delta_i \frac{\partial f_{i+1}}{\partial x_i} > 0, 1 \le i \le n-1$. For the system (1), John Smillie shows that every bound orbit of system (1) converges to an equilibrium. The system is a vital class of mathematical models in biology, and it can be used to model the ecosystems of n species $x_1, x_2, \cdots, x_n$ with a certain hierarchical structures.

In recent years, the dynamical behavior of the tridiagonal cooperative or competitive systems has received much attention, such as [2],[3],[4] and the references therein, and the results show that the long-term behaviour of the systems is relatively simple. For example, Smith [2] studied the nonlinear periodic tridiagonal system and showed that every bounded solution converges to the periodic solution. By using the appropriate coordinate transformation, the autonomous tridiagonal competitive system can be transformed into the autonomous tridiagonal cooperative system. Thus, without loss of generality, we only consider the cooperative case, i.e., $\delta_i = 1, 1 \le i \le n-1$. Generally, the discrete dynamical systems can have more complicated dynamics than the continuous systems. For instance, the discrete-time demographic model may exhibit chaotic dynamics [5]. Therefore, it is necessary to investigate the dynamical behaviour of the following discrete tridiagonal cooperative system:

$$\begin{cases} x_1^{(k)} = f_1(x_1^{(k-1)}, x_2^{(k-1)}), \\ x_j^{(k)} = f_j(x_{j-1}^{(k-1)}, x_j^{(k-1)}, x_{j+1}^{(k-1)}), 2 \leq j \leq n-1 \\ x_n^{(k)} = f_n(x_{n-1}^{(k-1)}, x_n^{(k-1)}). \end{cases} \tag{2}$$

It is worth noting that the property of the following linear system:

$$\begin{cases} x_1' = a_1 x_1 + b_1 x_2, \\ x_j' = c_{j-1} x_{j-1} + a_j x_j + b_{j+1} x_{j+1}, 2 \leq j \leq n-1, \\ x_n' = c_{n-1} x_{n-1} + a_n x_n \end{cases} \tag{3}$$

plays an important role in investigating the asymptotical behavior of system (1) and its corresponding continuous systems. In the system (3), $a_i > 0, i = 1, 2 \cdots, n, b_j > 0, c_j > 0, j = 1, 2, \cdots, n-1$. For the linear system (3), in [1] Smillie introduced an integer-valued Lyapunov function (see Section 2), and proved that along a nontrivial solution of the linear system the integer-valued Lyapunov function has the following properties [2]:

1. The Lyapunov function is defined for all t but an at most finite set of points;
2. The Lyapunov function is locally constant near points where it is defined;
3. The Lyapunov function strictly decreases as t increases through points where it is not defined.

By using the above properties of the integer-valued Lyapunov function for the nontrivial solution of the linear system, Smille [1], Smith[2], Wang [3] and Fang et al.[4] obtained the global asymptotical behavior of the corresponding tridiagonal competitive or cooperative systems. All the results show that the dynamics of the systems is relatively simple.

Due to the fact that the integer-valued Lyapunov function plays a vital role in the investigation of the global dynamics of system (1) and its corresponding systems, the integer-valued Lyapunov function may also play an important role in studying the global dynamics of the discrete tridiagonal cooperative system (2). Therefore, in this paper we attempt to explore the properties of the integer-valued Lyapunov function for the following linearized system of system (2):

$$\begin{cases} x_1^{(k)} = a_1 x_1^{(k-1)} + b_1 x_2^{(k-1)}, \\ x_j^{(k)} = c_{j-1} x_{j-1}^{(k-1)} + a_j x_j^{(k-1)} + b_{j+1} x_{j+1}^{(k-1)}, 2 \leq j \leq n-1, \\ x_n^{(k)} = c_{n-1} x_{n-1}^{(k-1)} + a_n x_n^{(k-1)}. \end{cases} \tag{4}$$

The main purpose of the paper is to study the property of the integer-valued Lyapunov function for system (4).

The paper is organized as follows：In section 2, we give the definition of the integer-valued Lyapunov function, and then present an example. The example shows that the property of the integer-valued Lyapunov function which holds true for the continuous case does not hold true for the discrete case. Thus we subsequently investigate a special class of the discrete tridiagonal cooperative linear systems, i.e., totally nonnegative matrix systems. Furthermore, in this section we also give the definitions of totally nonnegative matrix, oscillatory matrix and so on, and introduce the corresponding properties of these matrices. Then we state the main result of the paper in Section 2. The proof of the main result is given in Section 3. In Section 4, we conclude with a brief discussion on our theoretical and simulation results.

2 Main result

In order to investigate the property of the integer-valued Lyapunov function for the nontrivial solution of the system (4), it is necessary to give the definition of the integer-valued Lyapunov function. Let

$$\Lambda = \{v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in R^n : v_1 \neq 0, v_n \neq 0 \text{ and if } \quad v_i = 0 \text{ for some } i, 2 \leq i \leq n-1, \text{ then } v_{i-1}v_{i+1} < 0\}.$$

Let $v = (v_1, v_2, \cdots, v_n)^T$ be a vector in $\mathbb{R}^n$, all of whose coordinates v_i, $i = 1, 2, \cdots, n$, are nonzero. Define

$$\sigma(v) = \#\{i : v_i v_{i+1} < 0\},$$

where $\#$ denotes the cardinality of the set. It is easy to see that Λ is open and dense in $\mathbb{R}^n$. The map σ, called the integer-valued Lyapunov function, can be continuously extended to Λ, i.e., Λ is the maximal domain on which σ is continuous. For $x_0 \notin \Lambda$, $U(x_0, \delta)$ denotes the sufficiently small neighborhood with the center x_0. Define

$$\sigma_M(x_0) = \max_{x \in U(x_0,\delta) \cap \Lambda} \sigma(x).$$

It is obvious that $\sigma_M(x) = \sigma(x)$ if $x \in \Lambda$.

For ease of notation, let

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{pmatrix}, \tag{5}$$

where $a_i > 0, i = 1, 2, \cdots, n, b_i > 0, c_i > 0, i = 1, 2 \cdots, n-1$. It is easy to see that the solution to the system (4) can be expressed as

$$x^{(k)} = A^{k-1} x^{(1)},$$

where $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})^T, k = 1, 2, \cdots$. According to the corresponding result of the continuous system, we conjecture that the integer-valued Lyapunov function is also not increasing along the solution of system (4). We can easily see that the above conjecture is equivalent to the conjecture that $\sigma_M(Ax) \le \sigma(x)$ for all $x \in \Lambda$. Therefore, the purpose of the paper is to prove the following conjecture:

- **Conjecture:** $\sigma_M(Ax) \le \sigma(x)$ for all $x \in \Lambda$.

It is worth noting that the above conjecture does not hold true for all nonnegative tridiagonal matrices. For example, let

$$A = \begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 1 & 3 & 8 & 0 & 0 \\ 0 & 6 & 6 & 6 & 0 \\ 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 8 & 8 \end{pmatrix}; \qquad x = \begin{pmatrix} 8 \\ 10 \\ -5 \\ 6 \\ -5 \end{pmatrix}.$$

Direct computation yields that

$$Ax = \begin{pmatrix} 66 \\ -2 \\ 66 \\ -7 \\ 8 \end{pmatrix}.$$

It then follows that $\sigma(x) = 3, \sigma_M(Ax) = \sigma(Ax) = 4$. Thus $\sigma(x) < \sigma_M(Ax)$ which implies that the above conjecture does not hold true for all nonnegative tridiagonal matrices.

In the following we only consider the property of the integer-valued Lyapunov function for a class of nonnegative tridiagonal matrices, i.e., totally nonnegative tridiagonal matrices. Let A be an $n \times n$ matrix. Then we have:

Definition 1. The determinant of a $k \times k$ matrix obtained from A by deleting $n-k$ rows and $n-k$ columns is called a $k \times k$ **minor** of A (or **minor determinant of order** k of A).

Definition 2:[7] If all minors of any order for A are nonnegative (positive), the matrix A is called **totally nonnegative** (**totally positive**).

Definition 3:[7] If A is totally nonnegative and if there exists an integer $q > 0$ such that A^q is totally positive, A is called **oscillatory**.

For the nonnegative tridiagonal A defined in (5), we have the following result:

Lemma 1:[7] Let A be the nonnegative tridiagonal matrix defined in (5). Then A is oscillatory if and only if:

1. all the numbers $b_i, c_i, i = 1, 2, \cdots, n-1$ are positive;

2. the successive principal minors are positive, i.e.,

$$a_1 > 0, \begin{vmatrix} a_1 & b_1 \\ c_1 & a_2 \end{vmatrix} > 0, \begin{vmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{vmatrix} > 0, \cdots, \begin{vmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{vmatrix} > 0.$$

Now we are able to state the main result of the paper:

Theorem Let A be the nonnegative tridiagonal matrix defined in (5). If A is oscillatory, then $\sigma_M(Ax) \leq \sigma(x)$ for all $x \in \Lambda$.

3 Proof of the main result

In this section, we always assume that the matrix A represents the nonnegative tridiagonal matrix defined in (5) and A is oscillatory. In order to prove the main result of the paper, let us prove the following lemma.

Lemma 2. If $\sigma(Ay) \leq \sigma(y)$ for all y which satisfies that $y \in \Lambda$ and $Ay \in \Lambda$, then we have $\sigma_M(Ax) \leq \sigma(x)$ for all $x \in \Lambda$.

Proof. Since A is a nonsingular matrix and Λ is open in $\mathbb{R}^n$, it follows that for all $x_0 \in \Lambda$ there exists $\delta_1 > 0$ such that $V = A^{-1}U(Ax_0, \delta_1) \subset \Lambda$ and V is also open. In addition, since

$$\sigma_M(Ax_0) = \max_{y \in U(Ax_0, \delta_1) \cap \Lambda} \sigma(y),$$

it then follows that for all $y \in U(Ax_0, \delta_1) \cap \Lambda$ we have $\sigma_M(Ax_0) \geq \sigma(y)$. For all $y \in U(Ax_0, \delta_1) \cap \Lambda$ it is easy to see that there exists $x \in V$ such that $Ax = y$. Thus it follows from the definition of σ_M that there exists $y' \in U(Ax_0, \delta_1) \cap \Lambda$ and $x' \in V$ such that $\sigma_M(Ax_0) = \sigma(y'), Ax' = y'$. This implies that $\sigma_M(Ax_0) = \sigma(Ax')$. By using the given condition that $\sigma(Ax') \leq \sigma(x')$, we have $\sigma_M(Ax_0) \leq \sigma(x') = \sigma(x_0)$. This completes the proof of Lemma 2.

In order to prove the main result, it then follows from Lemma 2 that we only need to prove that $\sigma(Ax) \leq \sigma(x)$ for all x which satisfies that $x \in \Lambda$ and $Ax \in \Lambda$. In the following, we always assume that $x \in \Lambda$ and $Ax \in \Lambda$.

3.1 The proof for the case $n = 3$.

When $n = 3$, we have

$$A = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{pmatrix},$$

where $b_i > 0, c_i > 0, i = 1, 2$ and

$$a_1 > 0, \begin{vmatrix} a_1 & b_1 \\ c_1 & a_2 \end{vmatrix} > 0, \begin{vmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{vmatrix} > 0.$$

Let $x = (x_1, x_2, x_3)^T$. Then we have

$$Ax = \begin{pmatrix} a_1x_1 + b_1x_2 \\ c_1x_1 + a_2x_2 + b_2x_3 \\ c_2x_2 + a_3x_3 \end{pmatrix}.$$

In the following, let us consider the following three cases: (1)$\sigma(x) = 0$; (2)$\sigma(x) = 2$; (3)$\sigma(x) = 1$.

(1)Case $\sigma(x) = 0$. If $\sigma(x) = 0$, then the three elements in Ax have the same signs, i.e, $\sigma(Ax) = 0$. Thus the main result is proved.

(2)Case $\sigma(x) = 2$. Since $\sigma(x) = 2$, the number of changes of sign for the vector Ax is at most 2, i.e., $\sigma(Ax) \leq 2$. Thus the main result is proved.

(3)Case $\sigma(x) = 1$. Without loss of generality, we only discuss the case that $x_1 > 0, x_2 > 0, x_3 < 0$, and the other cases can be discussed similarly.

Let $x = (x_1, x_2, x_3)^T$, where $x_1 > 0, x_2 > 0, x_3 < 0$. Denote $y = Ax, y = (y_1, y_2, y_3)^T$. It is easy to see that $y_1 = a_1x_1 + b_1x_2 > 0$ and

$$\begin{cases} c_1x_1 + a_2x_2 + b_2x_3 = y_2, \\ c_2x_2 + a_3x_3 = y_3. \end{cases} \tag{6}$$

In the following we only prove that $y_2 < 0, y_3 > 0$ do not hold true. Suppose that $y_2 < 0, y_3 > 0$. Using the Cramer's Rule to solve Equ. (6) yields that

$$x_2 = \frac{\begin{vmatrix} y_2 - c_1x_1 & b_2 \\ y_3 & a_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ c_2 & a_3 \end{vmatrix}}; x_3 = \frac{\begin{vmatrix} a_2 & y_2 - c_1x_1 \\ c_2 & y_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ c_2 & a_3 \end{vmatrix}}.$$

Since the matrix A is oscillatory, it then follows that

$$\begin{vmatrix} a_2 & b_2 \\ c_2 & a_3 \end{vmatrix} > 0.$$

Because $a_2 > 0, a_3 > 0, b_2 > 0, c_1 > 0, c_2 > 0$ and $x_1 > 0, y_2 < 0, y_3 > 0$, we have $x_2 < 0, x_3 > 0$. This contradicts the fact that $x_2 > 0, x_3 < 0$. This contradiction implies that the main result is true for the case $n = 3$.

3.2 The proof for the case $n = 4$.

When $n = 4$,we have

$$A = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 \\ 0 & c_2 & a_3 & b_3 \\ 0 & 0 & c_3 & a_4 \end{pmatrix},$$

where $b_i > 0, c_i > 0, i = 1, 2, 3$ and

$$a_1 > 0, \begin{vmatrix} a_1 & b_1 \\ c_1 & a_2 \end{vmatrix} > 0, \begin{vmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{vmatrix} > 0, \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 \\ 0 & c_2 & a_3 & b_3 \\ 0 & 0 & c_3 & a_4 \end{vmatrix} > 0.$$

Let $x = (x_1, x_2, x_3, x_4)^T$. Then we have

$$Ax = \begin{pmatrix} a_1x_1 + b_1x_2 \\ c_1x_1 + a_2x_2 + b_2x_3 \\ c_2x_2 + a_3x_3 + b_2x_4 \\ c_3x_3 + a_4x_4 \end{pmatrix}.$$

In the following, let us consider four cases to prove the main result: (1)$\sigma(x) = 0$; (2)$\sigma(x) = 3$; (3)$\sigma(x) = 1$ and (4)$\sigma(x) = 2$.

(1)Case $\sigma(x) = 0$. Direct computation yields that all the elements in the vector Ax have the same signs, and it follows that $\sigma(Ax) = 0$. Thus in this case the main result is proved.

(2)Case $\sigma(x) = 3$. In this case, it is easy to see that the elements in Ax change the signs at most 3 times, i.e., $\sigma(Ax) \leq 3$. Thus in this case the result is proved.

(3)Case $\sigma(x) = 1$. Without loss of generality, we only discuss the case that $x_1 > 0, x_2 > 0, x_3 < 0, x_4 < 0$, and the other cases can be discussed similarly.

Let $x = (x_1, x_2, x_3, x_4)^T$, where $x_1 > 0, x_2 > 0, x_3 < 0, x_4 < 0$. Denote $y = Ax, y = (y_1, y_2, y_3, y_4)^T$. Direct computation yields that $y_1 = a_1x_1 + b_1x_2 > 0, y_4 = c_3x_3 + a_4x_4 < 0$ and

$$\begin{cases} c_1x_1 + a_2x_2 + b_2x_3 = y_2, \\ c_2x_2 + a_3x_3 + b_3x_4 = y_3. \end{cases} \tag{7}$$

In the following, we only need to prove that $y_2 < 0, y_3 > 0$ do not hold true. Suppose not. Then, by using the Cramer's Rule to solve Equ. (7), we have

$$x_2 = \frac{\begin{vmatrix} y_2 - c_1x_1 & b_2 \\ y_3 - b_3x_4 & a_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ c_2 & a_3 \end{vmatrix}}; x_3 = \frac{\begin{vmatrix} a_2 & y_2 - c_1x_1 \\ c_2 & y_3 - b_3x_4 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ c_2 & a_3 \end{vmatrix}}.$$

Since the matrix A is oscillatory, it then follows that

$$\begin{vmatrix} a_2 & b_2 \\ c_2 & a_3 \end{vmatrix} > 0.$$

Because $a_2 > 0, a_3 > 0, c_1 > 0, c_2 > 0$ and $x_1 > 0, y_2 < 0, x_4 < 0, y_3 > 0$ we have $x_2 < 0, x_3 > 0$. This contradicts the fact that $x_2 > 0, x_3 < 0$. This contradiction implies that the main result is true in this case.

(4)Case $\sigma(x) = 2$. Without loss of generality, we only discuss the case that $x_1 > 0, x_2 > 0, x_3 < 0, x_4 > 0$, and the other cases can be discussed similarly.

Let $x = (x_1, x_2, x_3, x_4)^T$, where $x_1 > 0, x_2 > 0, x_3 < 0, x_4 > 0$. Denote $y = Ax, y = (y_1, y_2, y_3, y_4)^T$. Direct computation yields that $y_1 = a_1x_1 + b_1x_2 > 0$ and

$$\begin{cases} c_1x_1 + a_2x_2 + b_2x_3 = y_2, \\ c_2x_2 + a_3x_3 + b_2x_4 = y_3, \\ c_3x_3 + a_4x_4 = y_4. \end{cases} \tag{8}$$

In the following, we only need to prove that $y_2 < 0, y_3 > 0, y_4 < 0$ do not hold true. Suppose not. By using the Cramer's Rule to solve Equ.(8), it follows that

$$\begin{aligned} x_2 &= \frac{(y_2 - c_1x_1)\Delta_{11} - y_3\Delta_{21} + y_4\Delta_{31}}{\Delta}; \\ x_3 &= \frac{-(y_2 - c_1x_1)\Delta_{12} + y_3\Delta_{22} - y_4\Delta_{32}}{\Delta}; \\ x_4 &= \frac{(y_2 - c_1x_1)\Delta_{13} - y_3\Delta_{23} + y_4\Delta_{33}}{\Delta}. \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a_2 & b_2 & 0 \\ c_2 & a_3 & b_4 \\ 0 & c_3 & a_4 \end{vmatrix},$$

and Δ_{ij} is the (i, j) minor of the determinant Δ.

Since A is oscillatory, it then follows that $\Delta > 0, \Delta_{ij} > 0, i, j = 1, 2, 3$. Because $c_1 > 0, x_1 > 0, y_2 < 0, y_3 > 0, y_4 < 0$, we have $x_2 < 0, x_3 > 0, x_4 < 0$. This contradicts the fact that $x_2 > 0, x_3 < 0, x_4 > 0$. This contradiction implies that the main result is true in this case.

In conclusion, the main result is proved for the case that $n = 4$.

3.3 The proof for the general case

In the previous two subsections, we provide the proof of the main theorem for the case that $n = 3$ and $n = 4$. In this subsection, we provide the proof of the main theorem for the general case.

Let $x = (x_1, x_2, \cdots, x_n)^T$ and $y = Ax$, where $y = (y_1, y_2, \cdots, y_n)^T, 1 \leq i < m \leq n$. Now let us state the following lemma which plays an important role in proving the main result.

Lemma 3. If $x_ix_{i+1} > 0, x_{m-1}x_m > 0$ and $x_jx_{j+1} < 0$ for all $j, i+1 \leq j \leq m-2$, then $x_ky_k < 0$ does not hold true for all $k, i+1 \leq k \leq m-1$.

Proof: Consider the equations from $(i+1)$-th to $(m-1)$-th in the equation $y = Ax$:

$$\begin{cases} c_i x_i + a_{i+1}x_{i+1} + b_{i+1}x_{i+2} = y_{i+1} \\ , c_{i+1}x_{i+1} + a_{i+2}x_{i+2} + b_{i+2}x_{i+3} = y_{i+2} \\ , c_{i+2}x_{i+2} + a_{i+3}x_{i+3} + b_{i+3}x_{i+4} = y_{i+3} \\ , \cdots\cdots \\ a_{m-3}x_{m-3} + a_{m-2}x_{m-2} + b_{m-2}x_{m-1} = y_{m-2} \\ , c_{m-2}x_{m-2} + a_{m-1}x_{m-1} + b_{m-1}x_m = y_{m-1}. \end{cases}$$

By using the Cramer's Rule to solve the above equations for $x_{i+1}, x_{i+2}, \cdots, x_{m-1}$, we have

$$\begin{aligned} x_j &= \frac{1}{\Delta}\left[\sum_{k=i+1}^{m-1} y_k(-1)^{k-i+j-i}\Delta_{k-i,j-i} - c_i x_i(-1)^{1+j-i}\Delta_{1,j-i} - \right. \\ &\qquad \left. b_{m-1}x_m(-1)^{m-i-1+j-i}\Delta_{m-i-1,j}\right], j = i+1, i+2, \cdots, m-1, \end{aligned} \tag{9}$$

where

$$\Delta = \begin{vmatrix} a_{i+1} & b_{i+1} & 0 & \cdots & 0 & 0 \\ c_{i+1} & a_{i+2} & b_{i+2} & \cdots & 0 & 0 \\ 0 & c_{i+2} & a_{i+3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{m-2} & b_{m-2} \\ 0 & 0 & 0 & \cdots & c_{m-2} & a_{m-1} \end{vmatrix},$$

and $\Delta_{i,j}, i, j = 1, 2, \cdots, m-i-1$ is the (i,j) minor of the determinant Δ.

Without loss of generality, we only consider the following case A, and the other cases can be proved similarly.

Case A: $x_i < 0, x_{i+1} < 0, x_{i+2} > 0, \cdots, x_{m-3} > 0, x_{m-2} < 0, x_{m-1} > 0, x_m > 0.$

In Case A, it is easy to see that $m-i+1$ is an even integer. In the following, we claim:

Assertion A: $y_{i+1} > 0, y_{i+2} < 0, \cdots, y_{m-3} < 0, y_{m-2} > 0, y_{m-1} < 0$ do not hold true for the case A.

Suppose that $y_{i+1} > 0, y_{i+2} < 0, \cdots, y_{m-3} < 0, y_{m-2} > 0, y_{m-1} < 0$. Since the matrix A is oscillatory, it then follows that $\Delta > 0, \Delta_{i,j} > 0, i, j = 1, 2, \cdots, m-i-1$. Because $c_i > 0, b_{m-1} > 0, x_i < 0, x_m > 0$, direct computations yield that

$$x_{i+1} = \frac{1}{\Delta}\left[\sum_{k=i+1}^{m-1} y_k(-1)^{k-i+1}\Delta_{k-i,1} - c_i x_i(-1)^2\Delta_{1,1} - b_{m-1}x_m(-1)^{m-i+2}\Delta_{m-i-1,i+1}\right] > 0.$$

By using the same arguments, it follows that $x_{i+2} < 0, \cdots, x_{m-3} < 0, x_{m-2} > 0, x_{m-1} < 0$. This contradicts the conditions in the case A. This contradiction implies that the Assertion A is proved under the conditions in Case A.

Now we are able to give the proof of the main theorem for the general case.

Proof of Theorem. Let $x = (x_1, x_2, \cdots, x_n)^T \in \Lambda$. If the vector x has zero elements, then according to the definition of Λ we can make sufficiently small perturbation for the zeros such that all the elements are non-zeros and the integer-valued Lyapunov function is not changed. Thus in the following we always assume that all elements in x are not zeros. For convenience, let us introduce the following notations.

1) The block in the vector x is called positive block of x, if the block in the vector x satisfies one of the following conditions:

(a) $x_{i-1} < 0,\ x_j > 0, j = i, i+1, \cdots, m, x_{m+1} < 0$ when $i > 1, m < n$, and denote $x_+^{i+1,m-1} = (x_{i+1}, x_{i+2}, \cdots, x_{m-1})^T$;

(b) $x_j > 0, j = 1, 2, \cdots, m, x_{m+1} < 0$ when $i = 1, m < n$, and denote $x_+^{1,m-1} = (x_1, x_2, \cdots, x_{m-1})^T$;

(c) $x_{i-1} < 0$,$x_j > 0, j = i, i+1, \cdots, n$ when $i > 1, m = n$,and denote $x_+^{i+1,n} = (x_{i+1}, x_{i+2}, \cdots, x_n)^T$;

(d) $x_j > 0, j = 1, 2, \cdots, n$ when $i = 1, m = n$, and denote $x_+^{1,n} = (x_{i+1}, x_{i+2}, \cdots, x_n)^T$;

2) The block in the vector x is called negative block of x, if the block in the vector x satisfies one of the following conditions:

(a) $x_{i-1} > 0, x_j < 0, j = i, i+1, \cdots, k, x_{m+1} > 0$ when $i > 1, m < n$, and denote $x_-^{i+1,k-1} = (x_{i+1}, x_{i+2}, \cdots, x_{m-1})^T$;

(b) $x_j < 0, j = 1, 2, \cdots, m, x_{m+1} > 0$ when $i = 1, m < n$, and denote $x_-^{1,m-1} = (x_1, x_2, \cdots, x_{m-1})^T$;

(c) $x_{i-1} > 0$,$x_j < 0, j = i, i+1, \cdots, n$ when $i > 1, m = n$, and denote $x_-^{i+1,n} = (x_{i+1}, x_{i+2}, \cdots, x_n)^T$;

(d) $x_j < 0, j = 1, i+1, \cdots, n$ when $i = 1, m = n$, and denote $x_-^{1,n} = (x_{i+1}, x_{i+2}, \cdots, x_n)^T$;

3)The block in the vector x is called sign-changing block of x, if the block in the vector x satisfies one of the following conditions:

(a) $x_{i-1}x_i > 0, x_j x_{j+1} < 0, j = i, i+1, \cdots, m, x_m x_{m+1} > 0$ when $i > 1, m < n$, and denote $x_\pm^{i,m} = (x_i, x_{i+2}, \cdots, x_m)^T$;

(b) $x_j x_{j+1} < 0, j = 1, 2, \cdots, m, x_m x_{m+1} > 0$ when $i = 1, m < n$, and denote $x_\pm^{1,m} = (x_1, x_2, \cdots, x_m)^T$;

(c) $x_{i-1}x_i > 0, x_j x_{j+1} < 0, j = i, i+1, \cdots, n-1$ when $i > 1, m = n$, and denote $x_\pm^{i,n} = (x_i, x_{i+2}, \cdots, x_n)^T$;

(d) $x_j x_{j+1} < 0, j = 1, 2, \cdots, n-1$ when $i = 1, m = n$, and denote $x_\pm^{1,n} = (x_1, x_2, \cdots, x_n)^T$;

According to the above notations, the vector x can be expressed as the following block vectors:

$$x = \begin{pmatrix} x^{[1]} \\ x^{[2]} \\ \cdots \\ x^{[k]} \end{pmatrix},$$

where $x^{[l]}, l = 1, 2, \cdots, k$ is positive, negative and sign-changing block of x. Denote $y = Ax$. By using the same manner to divide the vector y into blocks, and the vector y is expressed as

$$y = \begin{pmatrix} y^{[1]} \\ y^{[2]} \\ \cdots \\ y^{[k]} \end{pmatrix}.$$

We can asset that if $x^{[l]}$ is positive or negative block then $y^{[l]}$ is also positive or negative block, respectively. In the following, we only consider the case that $x^{[l]}$ is positive block, and the case that $x^{[l]}$ is negative block can also be discussed in a similar way. Without loss of generality, let $x^{[l]} = x_+^{i+1,m-1} = (x_{i+1}, x_{i+2}, \cdots, x_{m-1})^T$. Then we have $x_j > 0, j = i, i+1, \cdots, m$. Consider the equations from the $(i+1)$-th to the $(m-1)$-th in the expression $y = Ax$:

$$\begin{cases} c_i x_i + a_{i+1} x_{i+1} + b_{i+1} x_{i+2} = y_{i+1}, \\ c_{i+1} x_{i+1} + a_{i+2} x_{i+2} + b_{i+2} x_{i+3} = y_{i+2}, \\ c_{i+2} x_{i+2} + a_{i+3} x_{i+3} + b_{i+3} x_{i+4} = y_{i+3}, \\ \cdots\cdots \\ a_{m-3} x_{m-3} + a_{m-2} x_{m-2} + b_{m-2} x_{m-1} = y_{m-2}, \\ c_{m-2} x_{m-2} + a_{m-1} x_{m-1} + b_{m-1} x_m = y_{m-1}. \end{cases}$$

Because $x_j > 0, j = i, i+1, \cdots, m, a_i > 0, j = i+1, i+2, \cdots, m-1; c_j > 0, j = i, i+1, \cdots, m-2; b_j > 0, j = i+1, i+2, \cdots, m-1$, it then follows that $y_j > 0, j = i+1, i+2, \cdots, m-1$, i.e., $y^{[l]}$ is positive block.

Next, we only consider the case that the number of the blocks is 3, i.e., $k = 3$, and $x^{[1]} = x_\pm^{1,k_1}, x^{[2]} = x_+^{k_1+1,k_2-1}, x^{[3]} = x_\pm^{k_2,n}$, where $x_1 < 0, x_n > 0$. The other cases can be discussed in a similar way. Since $x^{[1]} = x_\pm^{1,k_1}, x^{[2]} = x_+^{k_1+1,k_2-1}, x^{[3]} = x_\pm^{k_2,n}$ and $x_1 < 0, x_m > 0$, it then follows that k_1 is an even number, $n - k_2 - 1$ is an odd number and $x_{k_1} > 0, x_{k_1+1} > 0, x_{k_2-1} > 0, x_{k_2} > 0$. By using the definition of σ we obtain that $\sigma(x) = k_1 - 1 + n - k_2$ and $\sigma(y^{[1]}) \le k_1 - 1$,$\sigma(y^{[3]}) \le n - k_2$. Thus it then follows from the above assertion that we have $\sigma(y^{[2]}) = 0$.

Because the dimension of the bolck $y^{[1]}$ is k_1, Lemma 3 implies that $\sigma(y^{[1]}) = k_1 - 1$ if and only if $sgn(y_j) = sgn(x_j), j = 1, 2, \cdots, k_1$. If $\sigma(y^{[1]}) = k_1 - 2$, then we have $y_1 > 0, sgn(y_j) = sgn(x_j), j = 2, 3, \cdots, k_1$ or $y_{k_1} < 0, sgn(y_j) = sgn(x_j), j = 1, 2, \cdots, k_1 - 1$. If $y^{[1]}$ has the other forms, we have $\sigma(y^{[1]}) \le k_1 - 3$. Similarly, $\sigma(y^{[3]}) = n - k_2$ if and only if $sgn(y_j) = sgn(x_j), j = k_2, k_2+1, \cdots, n$. If $\sigma(y^{[3]}) = n - k_2 - 1$, then we have $y_{k_2} < 0, sgn(y_j) = sgn(x_j), j = k_2+1, k_2+2, \cdots, n$ or $y_n < 0, sgn(y_j) = sgn(x_j), j = k_2, k_2+1, \cdots, n-1$.

If $y^{[3]}$ has the other forms, we have $\sigma(y^{[3]}) \leq n - k_2 - 2$. In the following, let us consider the following five cases:

Case 1) $\sigma(y^{[3]}) \leq n - k_2 - 2$ or $\sigma(y^{[1]}) \leq k_1 - 3$. Let us assume that $\sigma(y^{[3]}) \leq n - k_2 - 2$. If $\sigma(y^{[1]}) \leq k_1 - 3$, then we can discuss in a similar way. Direct computation yields that $\sigma(y) = \sigma(Ax) \leq \sigma(y^{[1]}) + \sigma(y^{[2]}) + \sigma(y^{[3]}) + 2 \leq k_1 - 1 + n - k_2 - 2 + 2 = \sigma(x)$. Thus the main theorem is proved in this case.

Case 2) $\sigma(y^{[1]}) = k_1 - 1$ and $\sigma(y^{[3]}) = n - k_2$. It then follows from the above discussion that $sgn(x_j) = sgn(y_j), j = 1, 2, \cdots, n$. Thus we have $\sigma(y) = \sigma(x)$, i.e., $\sigma(Ax) = \sigma(x)$. The main theorem is also proved in this case.

Case 3) $\sigma(y^{[1]}) = k_1 - 2$ and $\sigma(y^{[3]}) = n - k_2 - 1$. It then follows that $\sigma(y) = \sigma(Ax) \leq \sigma(y^{[1]}) + \sigma(y^{[2]}) + \sigma(y^{[3]}) + 2 \leq k_1 - 2 + n - k_2 - 1 + 2 = \sigma(x)$. The main theorem is proved in this case.

Case 4) $\sigma(y^{[1]}) = k_1 - 2, \sigma(y^{[3]}) = n - k_2$. It also follows from the above discussion that $sgn(x_j) = sgn(y_j), j = k_1 + 1, k_1 + 2, \cdots, n$. Thus we have $\sigma(y) = \sigma(Ax) \leq \sigma(y^{[1]}) + \sigma(y^{[2]}) + \sigma(y^{[3]}) + 1 \leq k_1 - 2 + n - k_2 + 1 = \sigma(x)$. The main theorem is then proved in this case.

Case 5) $\sigma(y^{[1]}) = k_1 - 1, \sigma(y^{[3]}) = n - k_2 - 1$, $\sigma(y^{[3]}) = n - k_2 - 1$. It follows from the above discussion that $sgn(x_j) = sgn(y_j), j = 1, 2, \cdots, k_2 - 1$. Thus we have $\sigma(y) = \sigma(Ax) \leq \sigma(y^{[1]}) + \sigma(y^{[2]}) + \sigma(y^{[3]}) + 1 \leq k_1 - 1 + n - k_2 - 1 + 1 = \sigma(x)$. The main theorem is then proved in this case.

In conclusion, the main theorem is completely proved.

4 Discussion

In this paper, we studied the property of an integer-valued Lyapunov function for a class of totally nonnegative matrices, and obtained that the integer-valued Lyapunov function for some vectors does not increase when the matrices act on the vectors. The properties may provide some preliminaries for studying the dynamics of some nonlinear discrete system, such as the system (2). Furthermore, we wrote MATLAB code to verify the validity of the result, and the code is attached in the Appendix. The simulation result showed that the main result obtained in this paper is true.

In the further studies on the question, by using the MATLAB code to run a lot of computations, the simulation result showed that:

Conjecture: If any minors of i-diagonal matrix is positive, then $\sigma(x) > \sigma_M(Ax)$ for all $x \in \Lambda$.

We guess that the above conjecture is true, and we leave this for further investigation.

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5 Appendix: Matlab code